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LETTER TO THE EDITOR

Canonical formulation of a generalized coupled dispersionless system

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Abstract. We give a canonical formulation of a generalized coupled dispersionless system proposed recently by the present authors. The symmetries of this system and their generators are given. A geometrical interpretation of this system is discussed.

1. Introduction

Inverse scattering schemes for soliton equations are a powerful tool for obtaining N-soliton solutions and an infinite number of conserved quantities. The most famous one is the ZS-AKNS scheme [1, 2]. Many inverse scattering schemes, such as the ZS-AKNS and its varieties, have 2×2 matrix forms. There are, however, fewer generalizations of them to 3×3 or higher-dimensional matrix forms. It is interesting to hunt for soliton equations with general $n \times n$ inverse scattering schemes. Recently, the present authors proposed a generalized coupled dispersionless system given by

$$S_{xt} + [S_x, [S, G]] = 0 \tag{1}$$

where the matrix field $S = S(x, t)$ and the constant matrix G are elements of an arbitrary non-Abelian Lie algebra [3]. This equation has the $n \times n$ ZS-AKNS-type inverse scattering scheme and nonlinearity comes from the non-Abelian character. Equation (1) is a generalization of the coupled integrable, dispersionless equations [6, 7]:

$$\begin{aligned} &q_{xt} + (rs)_x = 0 \\ &r_{xt} - 2q_x r = 0 \\ &s_{xt} - 2q_x s = 0 \end{aligned} \tag{2}$$

based on a group-theoretical point of view. For $SU(1,1) \sim O(2,1) \sim SL(2,R)$, equation (1) reproduces equation (2). For $SU(2) \sim O(3)$, we can obtain

$$\begin{aligned} &q_{xt} + (rr^*)_x = 0 \\ &r_{xt} - 2q_x r = 0 \\ &r^*_{xt} - 2q_x r^* = 0 \end{aligned} \tag{3}$$

which is equivalent to the Pohlmeyer–Lund–Regge system [4]. Equations (2) and (3) have been solved by the inverse scattering method under the appropriate boundary conditions and shown to be integrable [5–7]. They have the important conserved quantities

$$q_x^2 + r_x s_x = q_0^2 \tag{4}$$

and

$$q_x^2 + r_x r_x^* = q_0^2 \tag{5}$$

which are obtained from the inverse scattering schemes. Here $q_0 = q_x(\pm\infty)$ is constant.

The canonical formulation of the system (2) was given in [8] with the Dirac bracket [9]. In this letter, we present a canonical formulation of generalized coupled dispersionless system (1). In section 2 we shall review our general system. The canonical formulation is given in section 3 and in section 4 we shall consider the symmetries of the Lagrangian. Section 5 is devoted to discussions. There we also give a brief geometrical interpretation of the generalized dispersionless system and an analogy between this system and a particle in an external magnetic field.

2. The generalized coupled dispersionless system

Let $\mathcal{G}$ be a simple Lie group, $\dim \mathcal{G} = N$, and g be its Lie algebra. The generators T^a of g satisfy the commutation relation with the structure constant $f^{ab}{}_c$:

$$[T^a, T^b] = \mathrm{i} f^{ab}{}_c T^c \tag{6}$$

and the Cartan metric η^{ab} is defined by

$$\eta^{ab} = \mathrm{Tr}(T^a T^b). \tag{7}$$

Without loss of generality we can take η^{ab} as diagonal and $f^{ab}{}_c$ as totally anti-symmetric. With T^a's we define S by

$$S = \phi_a T^a = \eta_{ab}\phi^a T^b \tag{8}$$

where $\phi^a = \phi^a(x,t)$ is a vector field with components $(\phi^1, \phi^2, \ldots, \phi^N)$ and η_{ab} is the inverse matrix of η^{ab}. We also define a constant matrix G as

$$G = \kappa_a T^a \tag{9}$$

with a constant vector $\kappa^a = (\kappa^1, \kappa^2, \ldots, \kappa^N)$. These quantities are rotated by the global gauge transformation

$$S' = \Omega^{-1} S \Omega \tag{10}$$

$$G' = \Omega^{-1} G \Omega \tag{11}$$

where $\Omega \in \mathcal{G}$.

Let us write the action of generalized coupled dispersionless system as

$$I = \int \mathrm{d}t\ \mathrm{d}x\ \mathcal{L}(S, S_x, S_t) \tag{12}$$

where $\mathcal{L}$ is the Lagrangian density defined by

$$\mathcal{L} = \mathrm{Tr}\left(\tfrac{1}{2} S_x S_t - \tfrac{1}{3} G[S, [S_x, S]]\right). \tag{13}$$

This Lagrangian density is manifestly invariant under the global gauge transformation (10) and (11). The Euler–Lagrange equation can be derived from the action (12) by

$$S_{xt} - [[S, G], S_x] = 0. \tag{14}$$

From equation (14) it is easy to show that the following quantity:

$$\mathrm{Tr}(S_x^n) \tag{15}$$

is conserved for integer n $(n \geqslant 1)$ and is also invariant under gauge transformation.

This system has the following inverse scattering scheme:

$$\begin{aligned} V_x &= UV \\ V_t &= WV \end{aligned} \tag{16}$$

where the matrices U and W are elements of g and are defined with an eigenvalue λ as

$$\begin{aligned} U &= \lambda S_x \\ W &= W_0 + \frac{1}{\lambda} W_1 = [S, G] + \frac{1}{\lambda} G. \end{aligned} \tag{17}$$

This choice is an extension of the Kotlyarov's definition [4] for the $SU(2)$ to the general Lie algebra. The compatibility condition

$$U_t - W_x + [U, W] = 0 \tag{18}$$

then yields the equation of motion

$$S_{xt} - [W_0, S_x] = 0 \tag{19}$$

i.e. equation (14).

To obtain conservation law from the inverse scattering scheme, let the Jost function be

$$V_i = \exp(\Sigma_i). \tag{20}$$

Then, from the compatibility condition $V_{xt} = V_{tx}$, we get

$$(\Sigma_i)_{xt} = (\Sigma_i)_{tx}. \tag{21}$$

By using (16), equation (21) reduces to the form of the conservation law

$$\frac{\partial}{\partial t}\left(\sum_j U_{ij}\frac{V_j}{V_i}\right) = \frac{\partial}{\partial x}\left(\sum_j W_{ij}\frac{V_j}{V_i}\right). \tag{22}$$

Expanding V_j/V_i $(j \neq i)$ in the power series of $1/\lambda$ and equating the terms of the same power of $1/\lambda$, we can formally obtain an infinite number of conserved quantities [10].

Here we show some examples. For the case of $SU(1,1)$ we can classify according to $\kappa_a\kappa^a > 0$, $= 0$ and < 0. For $\kappa_a\kappa^a > 0$ equation (2) is obtained with $\phi^a = (q, \frac{1}{2}(r-s), \frac{1}{2}(r+s))$ and $\kappa^a = (1, 0, 0)$ and its inverse scattering scheme is given by

$$\begin{aligned} U &= \lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix} \\ W &= \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} - \frac{1}{2\lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{23}$$

The generators of $SU(1,1)$ are

$$T^1 = \frac{\tau^3}{\sqrt{2}} \qquad T^2 = \frac{\tau^1}{\sqrt{2}} \qquad T^3 = \frac{i\tau^2}{\sqrt{2}} \tag{24}$$

where the Pauli matrices τ^i are

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

The other choices of κ^a, say, $\kappa^a = (0, 0, 1)$ for $\kappa_a\kappa^a < 0$ and $\kappa^a = (1, 0, 1)$ for $\kappa_a\kappa^a = 0$ yield different equations.

For $SU(2)$, without loss of generality by virtue of the gauge invariance, we can choose $\phi^a = (\frac{1}{2}(r + r^*), (1/2\mathrm{i})(r - r^*), q)$ and $\kappa^a = (0, 0, 1)$ and the generators

$$T^1 = \frac{\tau^1}{\sqrt{2}} \qquad T^2 = \frac{\tau^2}{\sqrt{2}} \qquad T^3 = \frac{\tau^3}{\sqrt{2}} \tag{26}$$

and obtain equations (3). The inverse scheme is the same form as equations (25) with $s = r^*$.

When we choose another Lie algebra g with dimension N and appropriate κ^a, we obtain a dispersionless system with N-components having the inverse scattering scheme with a general $n \times n$ matrix.

3. Canonical formulation

In this section we present a canonical formulation of the generalized coupled dispersionless system (14). It is convenient to construct the canonical formulation in terms of ϕ^a. The Lagrangian density (13) becomes

$$\mathcal{L} = \tfrac{1}{2}\phi_x \cdot \phi_t + \tfrac{1}{3}\kappa \cdot [\phi \times (\phi_x \times \phi)] \tag{27}$$

where the symbol '$\cdot$' denotes the inner product defined by $\varphi \cdot \psi = \varphi^a\psi_a$ and '$\times$' the exterior product defined by $(\varphi \times \psi)_c = f^{ab}{}_c\varphi_a\psi_b$. The Euler–Lagrange equation is given by

$$\phi_{tx} + (\phi \times \kappa) \times \phi_x = 0 \tag{28}$$

through the variational principle. Linear independence of the basis T^a can also give (28) from (14). The canonical conjugate momentum is defined as

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \phi^a_t} = \frac{1}{2}\phi_{ax}. \tag{29}$$

It is obvious that the canonical conjugate momentum depends on ϕ^a. This is represented by the constraint equation

$$\chi^a \equiv \pi^a - \tfrac{1}{2}\phi^a_x = 0. \tag{30}$$

Since the usual Poisson bracket between the constraints does not vanish, this type of constraint is called a second-class constraint. According to Dirac, we introduce the Dirac bracket [9] defined by

$$\{F(x), G(y)\}_{\mathrm{DB}} \equiv \{F(x), G(y)\}_{\mathrm{PB}} - \int \mathrm{d}z\, \mathrm{d}z'\, \{F(x), \chi^a(z)\}_{\mathrm{PB}} \Delta^{-1}_{ab}(z - z')\{\chi^b(z'), G(y)\}_{\mathrm{PB}} \tag{31}$$

where the symbol $\{\cdot, \cdot\}_{\mathrm{PB}}$ stands for the Poisson bracket with the convention

$$\{\pi_a(x), \phi^b(y)\}_{\mathrm{PB}} = \delta^b_a\delta(x - y) \tag{32}$$

and Δ^{-1}_{ab} is the inverse of the matrix made from the Poisson bracket between the constraints

$$\Delta_{ab}(x - y) = \{\chi_a(x), \chi_b(y)\}_{\mathrm{PB}}. \tag{33}$$

The Dirac bracket (31) ensures that $\{\chi_a, \chi_b\}_{\mathrm{DB}} = 0$ and $\{R, \chi_b\}_{\mathrm{DB}} = 0$, where R is an arbitrary function of ϕ^a. Then, the fundamental Dirac bracket between the dynamical variables ϕ^a is

$$\{\phi^a(x), \phi^b(y)\}_{\mathrm{DB}} = \tfrac{1}{2}\eta^{ab}\,\mathrm{sgn}(x - y) \tag{34}$$

where $\mathrm{sgn}(x-y)$ is the sign function. This bracket relation is useful to calculate the relations between physical quantities.

According to the usual procedure, the Hamiltonian is given by

$$H = \int \mathrm{d}x\ \mathcal{H} \tag{35}$$

where $\mathcal{H}$ is the Hamiltonian density defined by

$$\begin{aligned}\mathcal{H} &= \phi_t \cdot \pi - \mathcal{L} \\ &= -\tfrac{1}{3}\kappa \cdot [\phi \times (\phi_x \times \phi)].\end{aligned} \tag{36}$$

Since the Hamiltonian is the generator of time translation, time evolution of the dynamical variables is given by the Hamilton equations in terms of the Dirac bracket as

$$\phi^a_t = \{H, \phi^a\}_{\mathrm{DB}} \tag{37}$$

$$\phi^a_{xt} = \{H, \phi^a_x\}_{\mathrm{DB}} \tag{38}$$

which coincide with the Euler–Lagrange equation (28).

The conserved quantity (15) for $n = 2$ becomes

$$\phi_x \cdot \phi_x \tag{39}$$

which reduces to (4) and (5) for $SU(1,1)$ and $SU(2)$, respectively.

4. Symmetries

The matrix formed Lagrangian density (13) is invariant under the global gauge transformation (10) and (11). Hence, it is obvious that the Lagrangian density (27) is also invariant under the infinitesimal version of the gauge transformation:

$$\delta\phi = \phi \times \theta \tag{40}$$

$$\delta\kappa = \kappa \times \theta \tag{41}$$

where θ^a is an infinitesimal constant parameter of the gauge transformation. Since κ^a is not a dynamical variable, the generators of the gauge transformation cannot obtained through the Noether theorem. However, we do have the generator of the gauge transformation (40) as

$$Q^a_G = \frac{1}{2}\int \mathrm{d}x\ (\phi_x \times \phi)^a. \tag{42}$$

Indeed, Q^a_G satisfies the Lie algebra

$$\{Q^a_G, Q^b_G\}_{\mathrm{DB}} = f^{ab}{}_c Q^c_G \tag{43}$$

and generates the gauge transformation

$$\{Q^a_G, \phi^b(x)\}_{\mathrm{DB}} = f^{ab}{}_c \phi^c. \tag{44}$$

Although the transformation (40) without (41) is not a symmetry of the Lagrangian density (27), there exists a special gauge transformation where $\delta\kappa = \kappa \times \theta = 0$. The Lagrangian density is really invariant under such a transformation and the generators of the special gauge transformation are conserved quantities.

The action is invariant under the 'reparametrization' of x:

$$\mathrm{d}x' = \mathrm{d}x + f_x(x)\,\mathrm{d}x \tag{45}$$

where $f(x)$ is an arbitrary infinitesimal function of x. As special cases, translation and scale transformation of x are included. The Noether theorem gives us the generator of the reparametrization

$$Q_R = \frac{1}{2}\int \mathrm{d}x\,(\phi_x \cdot \phi_x). \tag{46}$$

Although Q_R diverges because $\phi_x \cdot \phi_x = \text{constant}$, Q_R formally generates the correct transformation

$$\{Q_R, \phi^a(x)\}_{\mathrm{DB}} = \phi^a_x(x). \tag{47}$$

Q_R is the Casimir invariant of the Lie algebra g. In fact, the Dirac bracket satisfies the relation

$$\{Q^a_G, Q_R\}_{\mathrm{DB}} = 0 \tag{48}$$

i.e. Q_R is gauge invariant. The x-translational invariance means that $\frac{1}{2}\phi_x \cdot \phi_x$ is the momentum density of the field. The conservation law

$$(\phi_x \cdot \phi_x)_t = 0 \tag{49}$$

represents conservation of momentum density. Namely, the conserved quantity (15) has a physical meaning for $n = 2$. The action is also invariant under time translation, $t' = t + \epsilon$ where ϵ is a constant infinitesimal parameter and, of course, the Noether theorem yields the Hamiltonian (35).

5. Discussions

We have provided the canonical formulation of a generalized dispersionless system (28), i.e. equation (1). We have defined the Dirac bracket between the dynamical variables and presented the Hamiltonian and the Hamilton equations for the system. We have also given the generators of the symmetries of the action, including the global gauge symmetry and the reparametrization invariance of x. We have obtained the generators of the gauge transformation and found that the Lie algebra is realized by the Dirac bracket between the generators. From the invariance of the reparametrization, we have shown the total momentum Q_R of the field to be a conserved quantity.

Although the dispersionless system (1) has the inverse scattering scheme and has formally, but not explicitly, an infinite number of conserved quantities. The integrability of the generalized dispersionless system is not clear. This is still an open problem. We hope that our formulation is helpful in solving this problem.

Finally, we would like to give some comments on generalized coupled dispersionless systems. From the geometrical point of view it can be shown that the conserved quantity (15) or (39) is a natural consequence of the simple structure of the system (14) or (28). At a fixed time t, a solution S^0_x of (14) forms a certain curve in the space S_x. The equation of motion (14) describes the time evolution of this curve as

$$D_t S^0_x = 0 \tag{50}$$

where D_t is the covariant derivative defined as

$$D_t A = A_t - [W_0, A]. \tag{51}$$

Equation (50) may be interpreted as a parallel transport of each point x of the curve along the direction of t and W_0 plays the role of a connection. $\mathrm{Tr}((S^0_x)^2)$ is a conserved quantity because of (50). This situation is analogous to the square of the momentum of a charged particle interacting with an external magnetic field. Indeed, since the Lorentz force in the magnetic field is orthogonal to the momentum of the charged particle, the absolute value of the momentum is unchanged. So, assigning W_0 as a magnetic field, we can say that the coupled system (3) describes a current-conducting string with infinite length in a magnetic field for $O(3)$. Details will be discussed in a separate paper.

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